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Probability Theory and Applications (MA208)
Problem Sheet - 9

Some Important Continuous Random Variables

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1. Suppose that X has distribution $N(2, 0.16)$. Using the table of the normal distribution, evaluate the following probabilities.
   (a) $P(X \geq 2.3)$
   (b) $P(1.8 \leq X \leq 2.1)$
2. The diameter of an electric cable is normally distributed with mean 0.8 and variance 0.0004. What is the probability that the diameter will exceed 0.81 inch?
3. Suppose that the cable in Problem 9.2 is considered defective if the diameter differs from its mean by more than 0.025. What is the probability of obtaining a defective cable?
4. The errors in a certain length-measuring device are known to be normally distributed with expected value zero and standard deviation 1 inch. What is the probability that the error in measurement will be greater than 1 inch? 2 inches? 3 inches?
5. Suppose that the life lengths of two electronic devices, say D_1 and D_2, have distributions $N(40, 36)$ and $N(45, 9)$, respectively. If the electronic device is to be used for a 45-hour period, which device is to be preferred? If it is to be used for a 48-hour period, which device is to be preferred?
6. We may be interested only in the magnitude of X, say $Y = |X|$. If X has distribution $N(0, 1)$, determine the pdf of Y, and evaluate $E(Y)$ and $V(Y)$.
7. Suppose that we are measuring the position of an object in the plane. Let X and Y be the errors of measurement of the x- and y-coordinates, respectively. Assume that X and Y are independently and identically distributed, each with distribution $N(0, \sigma^2)$. Find the pdf of $R = \sqrt{X^2 + Y^2}$. (The distribution of R is known as the *Rayleigh distribution*.) [Hint: Let $X = R\cos\psi$ and $Y = R\sin\psi$. Obtain the joint pdf of (R, ψ) and then obtain the marginal pdf of R.]
8. Find the pdf of the random variable $Q = X/Y$, where X and Y are distributed as in Problem 9.7. (The distribution of Q is known as the *Cauchy* distribution.) Can you compute $E(Q)$?
9. A distribution closely related to the normal distribution is the *lognormal distribution*. Suppose that X is normally distributed with mean μ. and variance σ^2. Let $Y = e^x$. Then Y has the lognormal distribution. (That is, Y is lognormal if and only if InY is normal.) Find the pdf of Y. Note: The following random variables may be represented by the above distribution: the diameter of small particles after a crushing process, the size of an organism subject to a number of small impulses, and the life length of certain items.
10. Suppose that X has distribution $N(\mu, \sigma^2)$. Determine c (as a function of μ and σ) such that $P(X \leq c) = 2P(X > c)$.

11. Suppose that temperature (measured in degrees centigrade) is normally distributed with expectation 50° and variance 4. What is the probability that the temperature T will be between 48° and 53° centigrade?

12. The outside diameter of a shaft, say D, is specified to be 4 inches. Consider D to be a normally distributed random variable with mean 4 inches and variance 0.01 inch^2. If the actual diameter differs from the specified value by more than 0.05 inch but less than 0.08 inch, the loss to the manufacturer is \$0.50. If the actual diameter differs from the specified diameter by more than 0.08 inch, the loss is \$1.00. The loss, L, may be considered as a random variable. Find the probability distribution of L and evaluate $E(L)$.

13. Compare the *upper bound* on the probability $P[|X - E(X)| \geq 2\sqrt{V(X)}]$ obtained from Chebyshev's inequality with the exact probability in each of the following cases.

    (a) X has distribution $N(\mu, \sigma^2)$.

    (b) X has Poisson distribution with parameter λ.

    (c) X has exponential distribution with parameter α.

14. Suppose that X is a random variable for which $E(X) = \mu$. and $V(X) = \sigma^2$. Suppose that Y is uniformly distributed over the interval (a, b). Determine a and b so that $E(X) = E(Y)$ and $V(X) = V(Y)$.

15. Suppose that X, the breaking strength of rope (in pounds), has distribution $N(100, 16)$. Each 100-foot coil of rope brings a profit of \$25, provided $X > 95$. If $X \leq 95$, the rope may be used for a different purpose and a profit of \$10 per coil is realized. Find the expected profit per coil.

16. Let X_1 and X_2 be independent random variables each having distribution $N(\mu, \sigma^2)$. Let $Z(t) = X_1 \cos \omega t + X_2 \sin \omega t$. This random variable is of interest in the study of random signals. Let $V(t) = dZ(t)/dt$. (ω is assumed to be constant.)

    (a) What is the probability distribution of $Z(t)$ and $V(t)$ for any fixed t?

    (b) Show that $Z(t)$ and $V(t)$ are uncorrelated. [Note: One can actually show that $Z(t)$ and $V(t)$ are independent but this is somewhat more difficult to do.]

17. A rocket fuel is to contain a certain percent (say X) of a particular compound. The specifications call for X to be between 30 and 35 percent. The manufacturer will make a net profit on the fuel (per gallon) which is the following function of X:

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\begin{aligned}
T(X) &= \$0.10 \text{ per gallon} \quad \text{if } 30 < X < 35, \\
&= \$0.05 \text{ per gallon} \quad \text{if } 35 \leq X < 40 \text{ or } 25 < X \leq 30, \\
&= -\$0.10 \text{ per gallon otherwise.}
\end{aligned}
$$

    (a) If X has distribution $N(33, 9)$, evaluate $E(T)$.

    (b) Suppose that the manufacturer wants to increase his expected profit, $E(T)$, by 50 percent. He intends to do this by increasing his profit (per gallon) on those batches of fuel meeting the specifications, $30 < X < 35$. What must his new net profit be?

18. Consider Example 9.8. Suppose that the operator is paid C_3 dollars/hour while the machine is operating and C_4 dollars/hour ($C_4 < C_3$) for the remaining time he has been hired after the machine has failed. Again determine for what value of H (the number of hours the operator is being hired), the expected profit is maximized.

19. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (See 9.15.) [Hint: Make the change of variable $x = \mu^2/2$ in the integral $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$.]

20. Verify the expressions for $E(X)$ and $V(X)$, where X has a Gamma distribution [see Eq. (9.18)].

21. Prove Theorem 9.3.

22. Prove Theorem 9.4.

23. Suppose that the random variable X has a chi-square distribution with 10 degrees of freedom. If we are asked to find two numbers a and b such that $P(a < x < b) = 0.85$, say, we should realize that there are many pairs of this kind.

    (a) Find two different sets of values (a, b) satisfying the above condition.

    (b) Suppose that in addition to the above, we require that

    $$P(X < a) = P(X > b).$$

    How many sets of values are there?

24. Suppose that V, the velocity (cm/sec) of an object having a mass of 1 kg, is a random variable having distribution $N(0, 25)$. Let $K = 1000V^2/2 = 500V^2$ represent the kinetic energy (KE) of the object. Evaluate $P(K < 200), P(K > 800)$.

25. Suppose that X has distribution $N(\mu, \sigma^2)$. Using Theorem 7.7, obtain an approximation expression for $E(Y)$ and $V(Y)$ if $Y = In\ X$.

26. Suppose that X has a normal distribution truncated to the right as given by Eq. (9.22). Find an expression for $E(X)$ in terms of tabulated functions.

27. Suppose that X has an exponential distribution truncated to the left as given by Eq. (9.24). Obtain $E(X)$.

28. (a) Find the probability distribution of a binomially distributed random variable (based on n repetitions of an experiment) truncated to the right at $X = n$; that is, $X = n$ cannot be observed.

    (b) Find the expected value and variance of the random variable described in (a).

29. Suppose that a normally distributed random variable with expected value μ and variance σ^2 is truncated to the left at $X = \tau$ and to the right at $X = \gamma$. Find the pdf of this "doubly truncated" random variable.

30. Suppose that X, the length of a rod, has distribution $N(10, 2)$. Instead of measuring the value of X, it is only specified whether certain requirements are met. Specifically, each manufactured rod is classified as follows: $X < 8, 8 \leq X < 12$, and $X \geq 12$. If 15 such rods are manufactured, what is the probability that an equal number of rods fall into each of the above categories?

31. The annual rainfall at a certain locality is known to be a normally distributed random variable with mean value equal to 29.5 inches and standard deviation 2.5 inches. How many inches of rain (annually) is exceeded about 5 percent of the time?

32. Suppose that X has distribution $N(0, 25)$. Evaluate $P(1 < X^2 < 4)$.

33. Let X_t be the number of particles emitted in t hours from a radioactive source and suppose that X_t has a Poisson distribution with parameter βt. Let T equal the number of hours until the first emission. Show that T has an exponential distribution with parameterβ. [Hint: Find the equivalent event (in terms of X_t,) to the event $T > t$.]

34. Suppose that X_t is defined as in Problem 9.33 with $\beta = 30$. What is the probability that the time between successive emissions will be $>$ 5 minutes? $>$ 10 minutes? $<$ 30 seconds?

35. In some tables for the normal distribution, $H(x) = (1/\sqrt{2\pi}) \int_0^x e^{-t^2/2} dt$ is tabulated for positive values of x (instead of$\Phi(x)$ as given in the Appendix). If the random variable X has distribution $N(1,4)$ express each of the following probabilities in terms of *tabulated* values of the function H.

    (a) $P[|X| > 2]$
    (b) $P[X < 0]$

36. Suppose that a satellite telemarketing device receives two kinds of signals which may be recorded as real numbers, say X and Y. Assume that X and Y are independent, continuous random variables with pdf's f and g, respectively. Suppose that during any specified period of time only one of these signals may be received and hence transmitted back to earth, namely that signal which arrives first. Assume furthermore that the signal giving rise to the value of X arrives first with probability p and hence the signal giving rise to Y arrives first with probability $1 - p$. Let Z denote the random variable whose value is actually received and transmitted.

    (a) Express the pdf of Z in terms of f and g.
    (b) Express $E(Z)$ in terms of $E(X)$ and $E(Y)$.
    (c) Express $V(Z)$ in terms of $V(X)$ and $V(Y)$.
    (d) Suppose that X has distribution $N(2,4)$ and that Y has distribution $N(3,3)$. If $p = \frac{2}{3}$, evaluate $P(Z > 2)$.
    (e) Suppose that X and Y have distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Show that if $\mu_1 = \mu_2$, the distribution of Z is "uni-modal," that is, the pdf of Z has a unique relative maximum.

37. Assume that the number of accidents in a factory may be represented by a Poisson process averaging 2 accidents per week. What is the probability that (a) the time from one accident to the next will be more than 3 days, (b) the time from one accident to the third accident will be more than a week? [Hint: In (a), let $T =$ time (in days) and compute $P(T > 3)$.]

38. On the average a production process produces one defective item among every 300 manufactured. What is the probability that the third defective item will appear:

    (a) before 1000 pieces have been produced?
    (b) as the 1000th piece is produced?
    (c) after the 1000th piece has been produced?

    [Hint: Assume a Poisson process.]

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